Ch 1. First - order ODEs.

1° (1-4)

看到 $M(x,y)\,dx + N(x,y)\,dy = 0$：

① check：$\frac{\partial M}{\partial y} \overset{?}{=} \frac{\partial N}{\partial x}$

是 → $y = \int M dx + \int N dy + k(y) + l(x)$

否 → $FMdx + FNdy = 0$，找到對應的 F → $y = \int FMdx + \int FNdy + k(y) + l(x)$

Q：如何找 F？

Ans：F 可以是 $F(x)$ 或 $F(y)$，

→ $F(x) = e^{\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) dx}$

→ $F(y) = e^{\int \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy}$

看 $F(x)$ 和 $F(y)$ 哪个型式比較簡單，選擇用簡單的

選題．10.

2° (1-5)

看到 $y' + P(x)y = 0$ ：$y = \text{constant} \cdot e^{-\int P(x)dx}$

看到 $y' + P(x)y = r(x)$ ：$y = e^{-\int P(x)dx} \cdot \int r(x) e^{\int P(x)dx} dx + \text{constant} \cdot e^{-\int P(x)dx}$

看到 $y' + P(x)y = g(x)y^a$：Let $u'(x) = (1-a)(y(x))^{-a} y'(x)$ → $u'(x) + (1-a)P(x)u(x) = (1-a)g(x)$

→ 求出 u 再由 u 求 y

選題．23.

Ch2、（概念沒問題，應用題有問題） Second order ODEs

1° If $y''+ay'+by=0$:

(2.2)

$\to$ solve $\lambda^2+a\lambda+b=0$,

- If $\lambda=\alpha,\beta$, $\alpha\neq\beta$ & $\alpha,\beta\in\mathbb{R}$: $y=c_1e^{\alpha x}+c_2e^{\beta x}$
- If $\lambda=\alpha$, $\alpha\in\mathbb{R}$: $y=c_1e^{\alpha x}+c_2xe^{\alpha x}$
- If $\lambda=\alpha\pm i\beta$ (complex) : $y=e^{\alpha x}(c_1\sin\beta x+c_2\cos\beta x)$

2° If $x^2y''+axy'+by=0$ (Euler-Cauchy equations) :

Let $y=x^m$, $y'=mx^{m-1}$, $y''=m(m-1)x^{m-2}$

$\to$ $m(m-1)+am+b=0$, $m^2+(a-1)m+b=0$, $m=\frac{(1-a)\pm\sqrt{(a-1)^2-4b}}{2}$

$\to$
- If $m=\alpha,\beta$, $\alpha,\beta\in\mathbb{R}$ & $\alpha\neq\beta$: $y=c_1x^{\alpha}+c_2x^{\beta}$
- If $m=\alpha$, $\alpha\in\mathbb{R}$: $y=(c_1+c_2\ln x)x^m$
- If $m=\alpha\pm i\beta$: $y=x^{\alpha}(c_1\cos\beta\ln x+c_2\sin\beta\ln x)$

選題： 2-4 9、

額外題： 1-4-5、 1-3-24 ， 1-5-40、 2-5-14

1-1

5.

$y' = 4e^{-x}\cos x$

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$

$y = 4\int e^{-x}\cos x\,dx$

$= 4\int e^{-x}\cdot\frac{e^{ix}+e^{-ix}}{2}\,dx$

$= 2\int e^{(i-1)x} + e^{(-i-1)x}\,dx$

$= 2\left[\frac{1}{i-1}e^{(i-1)x} - \frac{1}{i+1}e^{(-i-1)x}\right] + C$

$= 2e^{-x}\left[\frac{e^{ix}}{i-1} - \frac{e^{-ix}}{i+1}\right] + C$

$= 2e^{-x}\left[\frac{(\cancel{i}+1)e^{ix} - \cancel{i}(i-1)e^{-ix}}{(i-1)(i+1)}\right] + C$

$= 2e^{-x}\left[\frac{e^{ix}+e^{-ix}}{-1-1}\right] + C$

$= -2e^{-x}\cos x$

6. $y'' = -y$, $\frac{d^2y}{dx^2} = -y$, Let $y = e^{kx}$, $\frac{d^2y}{dx^2} = k^2e^{kx} = k^2y$, $k = \pm i$

$\rightarrow y = A_1e^{ix} + A_2e^{-ix}$

16.

$y = cx - c^2$, $y' = c$

$\rightarrow \cancel{c^2} - \cancel{xc} + \cancel{cx} - \cancel{c^2} = 0$

1.3

8.

$y' = (y+4x)^2 \qquad y+4x = v, \quad y'+4 = v'$

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$v' - 4 = v^2, \quad \frac{dv}{dx} = v^2+4, \quad \frac{1}{v^2+4}\,dv = dx, \quad \int \frac{1}{v^2+4}\,dv = x+C$

$y = \tan^{-1}x, \quad \tan y = x, \quad \sec^2 y\,\frac{dy}{dx} = 1, \quad \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{\tan^2 y + 1} = \frac{1}{x^2+1}$

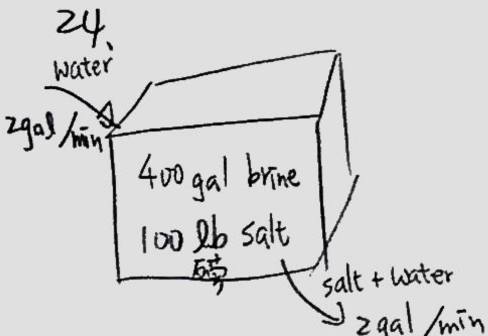
$\int \frac{1}{v^2+4}\,dv = x+C, \quad \frac{1}{4}\int \frac{1}{\frac{v^2}{4}+1}\,dv = x+C \qquad$ let $\frac{v}{2} = u, \; dv = 2du$

$2\cdot\frac{1}{4}\int \frac{1}{u^2+1}\,du = x+C, \quad \frac{1}{2}\tan^{-1}u = x+C, \quad \frac{1}{2}\tan^{-1}\frac{v}{2} = x+C,$

$\frac{1}{2}\tan^{-1}\frac{y+4x}{2} = x+C, \quad \frac{1}{2}\tan^{-1}\left(\frac{y}{2}+2x\right) = x+C, \quad \tan^{-1}\left(\frac{y}{2}+2x\right) = 2x+C$

$\frac{y}{2}+2x = \tan(2x+C), \qquad y = 2\tan(2x+C) - 4x$

24.



$S(t), \quad \underset{\downarrow\, d}{\Delta S} = -2\underset{\downarrow\, d}{\Delta t}\left(\frac{S}{400}\right) \qquad dS = -2dt\left(\frac{S}{400}\right) \qquad \frac{dS}{dt} = -\frac{S}{200}, \quad S(t) = \overset{100\,lb}{\overset{\parallel}{S(0)}}\, e^{-\frac{t}{200}}$

$S(t) = 100\,e^{-\frac{t}{200}}. \quad S(60\,\text{min}) = 100\,e^{-\frac{60}{200}} = 100\,e^{-0.3}$

1-4. Integrating Factors.

$$du = \underbrace{\boxed{\frac{\partial u}{\partial x}}}_{M} dx + \underbrace{\boxed{\frac{\partial u}{\partial y}}}_{N} dy$$

$M(x,y)\,dx + N(x,y)\,dy = 0$ du, u. constant

$\rightarrow \frac{\partial u}{\partial x} = M$ & $\frac{\partial u}{\partial y} = N$

$$\frac{\partial^2 u}{\partial x \partial y} = \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}} \qquad u = \int M dx + k(y) + \int N dy + J(x)$$

But Not works' for all ODEs,

$\rightarrow$ Need to find Intergrating Factors.

If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$:

$M(x,y)\,dx + N(x,y)\,dy = 0$

$FM\,dx + FN\,dy = 0$

$$\frac{\partial F}{\partial y} M + F\frac{\partial M}{\partial y} = \frac{\partial F}{\partial x} N + F\frac{\partial N}{\partial x}$$

Let $F = F(x)$ (indep. of y) (Just for simpler)

$$F\frac{\partial M}{\partial y} = \frac{\partial F}{\partial x} N + F\frac{\partial N}{\partial x} \qquad \div FN$$

$$\frac{1}{N}\frac{\partial M}{\partial y} = \frac{1}{F}\frac{\partial F}{\partial x} + \frac{1}{N}\frac{\partial N}{\partial x}$$

$$\frac{1}{F}\frac{\partial F}{\partial x} = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = R$$

① If $R = R(x)$

$$\frac{1}{F}\frac{dF}{dx} = R, \quad \frac{dF}{dx} = FR, \quad F = e^{\int R\,dx}$$

② If $R = R(y)$

$$\frac{1}{F}\frac{dF}{dy} = R, \quad \frac{dF}{dy} = FR, \quad F = e^{\int R\,dy}, \quad R = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

1-4

5、

$(x^2+y^2)dx - 2xy\,dy = 0$

$M: x^2+y^2, \quad \frac{\partial M}{\partial x} = 2x$

$N = -2xy, \quad \frac{\partial N}{\partial y} = -2x$

$R = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)_{(F(x))}$ or $R = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)_{(F(y))}$

$= -\frac{1}{2xy}\left(2y-(-2y)\right) = -\frac{4y}{2xy} = -\frac{2}{x}$

$F = e^{\int R dx} = e^{-\int \frac{2}{x}dx} = e^{-2\ln x} = e^{\ln \frac{1}{x^2}} = \frac{1}{x^2}$

$\frac{1}{x^2}(x^2+y^2)dx - \frac{1}{x^2}(2xy)\,dy = 0$

$\left(1+\frac{y^2}{x^2}\right)dx - \frac{2y}{x}dy = 0 = du$

$\left.\begin{array}{ll} \frac{\partial u}{\partial x} = 1+\frac{y^2}{x^2}, & u = x - \frac{y^2}{x} + k(y) \\ \frac{\partial u}{\partial y} = -\frac{2y}{x}, & u = -\frac{y^2}{x} + I(x) \end{array}\right] \rightarrow u = x - \frac{y^2}{x} + \text{constant}.$

$du = 0$, the solution is $x - \frac{y^2}{x} = \text{constant}$,

10、

$y\,dx + (y+\tan(x+y))\,dy = 0$, $\cos(x+y)$

$\underbrace{y(\cos(x+y))}_{M}dx + (\underbrace{y\cos(x+y)+\sin(x+y)}_{N})\,dy = 0$

$\frac{\partial M}{\partial y} = \cos(x+y) - y\sin(x+y)$; $\frac{\partial N}{\partial x} = -y\cos(x+y) + \cos(x+y)$

$\frac{\partial u}{\partial x} = y\cos(x+y)$, $u = y\sin(x+y) + k(y)$

$\frac{\partial u}{\partial y} = y\cos(x+y)+\sin(x+y)$, $u = y\sin(x+y) + I(x)$

$du = 0$, $u = \text{constant}$.

solution : $y\sin(x+y) = \text{constant}$.

1-5.

① If $y' + p(x)y = 0$, $y' = -p(x)y$, $y = \overset{\text{complex}}{c}\, e^{-\int p(x)dx}$ //

② If
$y' + p(x)y = r(x)$, $Fy' + pFy = rF$ $\quad (Fy)' = F'y + pFy = rF$

If $pFy = F'y$, $(Fy)' = rF$

$\Downarrow$

$pF = F'$

$p = \frac{1}{F}\frac{dF}{dx}$ $\qquad F = e^h$

$\int p\,dx = \int \frac{1}{F} dF = \ln|F| = h$, $h' = p$

$Fy' + FPy = rF$, $e^h y' + e^h h' y = re^h$, $(e^h y)' = re^h$

$e^h y = \int re^h dx + c$

$y = e^{-h}\int re^h dx + ce^{-h}$

$= e^{-\int p dx}\int re^{\int p dx} dx + c\,e^{-\int p dx}$

③ If $y' + P(x)y = g(x)y^a$, set $u(x) = [y(x)]^{1-a}$, $u'(x) = (1-a)[y(x)]^{-a}y'(x)$

$\frac{1}{1-a}\, y(x)^a u'(x) + P(x)y = g(x)y^a$

$\frac{1}{1-a}\, y^a u' = g(x)y^a - py$

$u' = (1-a)y^{-a}(g(x)y^a - py) = (1-a)g(x) - (1-a)py^{-a}$

$u' + (1-a)pu = (1-a)g$

1-5.

7.

$xy' = 2y + x^3 e^x$, $p(x) = -\frac{2}{x}$ $r(x)$

$y' = \frac{2}{x} y + x^2 e^x$. $y' - \frac{2}{x} y = x^2 e^x$

$y = e^{-\int p\,dx} \int r e^{\int p\,dx}\, dx + C e^{-\int p\,dx}$

$\int p\,dx = \int -\frac{2}{x}\, dx = -2 \ln x$, $e^{-2\ln x} = e^{\ln \frac{1}{x^2}} = \frac{1}{x^2}$, $e^{-\int p\,dx} = x^2$

$y = x^2 \int r \cdot \frac{1}{x^2} \cdot dx + C x^2$

$= x^2 \int e^x\, dx + C x^2 = x^2 e^x + \text{constant} \cdot x^2 + \text{constant}$

23.

$y' + xy = \frac{x}{y}$, $y(0) = 3$ Let $u(x) = y^2$, $u' = 2yy'$

$\frac{u'}{2y} + xy = \frac{x}{y}$

$u' + 2xy^2 = 2x$, $u' = 2x - 2xu$, $\frac{du}{dx} = 2x(1-u)$, $\frac{1}{1-u}\, du = 2x\, dx$

$\int \frac{1}{1-u}\, du = \int 2x\, dx$, $\ln \frac{1}{|1-u|} = x^2 + \text{constant}$,

$\ln \frac{1}{|1-y^2|} = x^2 + \text{constant}$, $\frac{1}{|1-y^2|} = e^{x^2 + \text{constant}} = \text{constant} \cdot e^{x^2}$

$\frac{1}{y^2 - 1} = \text{constant} \cdot e^{x^2}$, $y^2 - 1 = \text{constant} \cdot e^{-x^2}$, $y^2 = \text{constant} \cdot e^{-x^2} + 1$

$x = 0$, $y(0) = 3$

↳ $y(0)^2 = \text{constant} + 1 = 9$, constant $= 8$

→ $y^2 = 8e^{-x^2} + 1$.

27.

$$y' = \frac{1}{6e^y - 2x}$$

$$6y'e^y - 2xy' = 1$$

$$6e^y \frac{dy}{dx} - 2x\frac{dy}{dx} = 1$$

$$(6e^y - 2x)\,dy - dx = 0$$

$$-dx + (6e^y - 2x)\,dy = 0$$

$$M = -1, \quad N = 6e^y - 2x$$

$$R = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{6e^y - 2x}(2)$$

or

$$R = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = -1(-2) = 2 \quad (V).$$

$$F = e^{\int R\,dy} = e^{\int 2\,dy} = e^{2y}$$

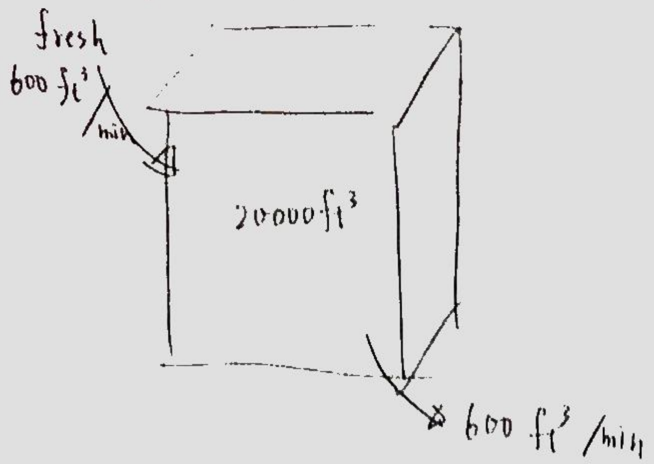
$$-e^{2y}\,dx + (6e^{3y} - 2xe^{2y})\,dy = 0$$

$$\frac{\partial u}{\partial x} = -e^{2y}, \quad u = -xe^{2y} + k(y)$$

$$\frac{\partial u}{\partial y} = 6e^{3y} - 2xe^{2y}, \quad u = 2e^{3y} - xe^{2y} + I(x)$$

$$\to \quad u = \boxed{-xe^{2y} + 2e^{3y} = \text{constant}}$$

40.



$$\Delta F(t)_{min} = 600\Delta t_{min} - \frac{F(t)}{\cancel{20000}_{100}} \times \cancel{600}^{3}\Delta t$$

$$dF = 600\,dt - 0.03F\,dt$$

$$\frac{dF}{dt} = 600 - 0.03F$$

$$F' + 0.03F = 600$$

Let $F = A_1 e^{-0.03t} + A_2$

$$F' = -0.03A_1 e^{-0.03t}$$

$$+)\ 0.03F = 0.03A_1 e^{-0.03t} + 0.03A_2$$

$$F' + 0.03F = 0.03A_2 = 600, \quad \frac{600}{0.03} = A_2, \quad A_2 = 20000$$

$$F = A_1 e^{-0.03t} + 20000$$

$$F(0) = 0, \quad A_1 + 20000 = 0, \quad A_1 = -20000$$

$$F = -20000e^{-0.03t} + 20000$$

When 90%,

$$\frac{\cancel{20000}e^{-0.03t}}{\cancel{20000}} = 0.1, \quad e^{-0.03t} = 0.1, \quad -0.03t = -\ln 10$$

$$0.03t = \ln 10, \quad t = \frac{100}{3}\ln 10$$

$(\sim 77\,min)$.

2-1.

・二階 ODE 可由二組 ODE 解的線性組合構成

5、

$yy'' = 3y'^2$ Let $u = y' = \frac{dy}{dx}$, $\frac{du}{dx} = \frac{dy}{dx}\frac{du}{dy} = u\frac{du}{dy}$

$yu\frac{du}{dy} = 3u^2$, $y\frac{du}{dy} = 3u$, $y\,du = 3u\,dy$, $\frac{1}{3u}du = \frac{1}{y}dy$,

$\int \frac{1}{3u}du = \int \frac{1}{y}dy + \text{constant}$, $\frac{1}{3}\ln|u| = \ln|y| + \text{constant}.$

$\frac{1}{3}\ln\frac{dy}{dx} = \ln y + \text{constant}$. $\left(\frac{dy}{dx}\right)^{1/3} = \text{constant}\cdot y$

$\frac{dy}{dx} = \text{constant}\cdot y^3$, $\int \frac{1}{y^3}dy = \int \text{constant}\cdot dx$

$-\frac{1}{2y^2} = \text{constant}\cdot x + \text{constant}$,

$\text{constant}\cdot xy^2 + \text{constant}\cdot y^2 = 1$ #

6、

$xy'' + 2y' + xy = 0$, $y_1 = \frac{\cos x}{x}$

Check y_1: $y_1' = \frac{-x\sin x - \cos x}{x^2} = -\frac{\sin x}{x} - \frac{\cos x}{x^2}$

$y'' = -\left(\frac{x\cos x - \sin x}{x^2}\right) - \left(\frac{-x^2\sin x - 2x\cos x}{x^4}\right)$

$= -\frac{\cos x}{x} + \frac{2\sin x}{x^2} + \frac{2\cos x}{x^3}$

$x\left(\cancel{-\frac{\cos x}{x}} + \cancel{\frac{2\sin x}{x^2}} + \cancel{\frac{2\cos x}{x^3}}\right) + 2\left(\cancel{-\frac{\sin x}{x}} - \cancel{\frac{\cos x}{x^2}}\right) + x\cdot\cancel{\frac{\cos x}{x}} = 0$ (V)

猜 $y_2 = \frac{\sin x}{x}$

check : $y_2' = \frac{x\cos x - \sin x}{x^2} = \frac{\cos x}{x} - \frac{\sin x}{x^2}$

$y_2'' = \frac{-x\sin x - \cos x}{x^2} - \frac{x^2\cos x - 2x\sin x}{x^4} = \frac{-\sin x}{x} - \frac{2\cos x}{x^2} + \frac{2\sin x}{x^3}$

$x\left(\cancel{-\frac{\sin x}{x}} - \cancel{\frac{2\cos x}{x^2}} + \cancel{\frac{2\sin x}{x^3}}\right) + 2\left(\cancel{\frac{\cos x}{x}} - \cancel{\frac{\sin x}{x^2}}\right) + x\cdot\cancel{\frac{\sin x}{x}} = 0$ (V)

$y = A\frac{\cos x}{x} + B\frac{\sin x}{x}.$

2-2.

3、

$y'' + 4y' + 2.5y = 0$

$\lambda^2 + 4\lambda + 2.5 = 0$

$\lambda = \frac{-4 \pm \sqrt{16-10}}{2} = -2 \pm \frac{1}{2}\sqrt{6}$

$y = Ae^{(-2+\frac{\sqrt{6}}{2})x} + Be^{(-2-\frac{\sqrt{6}}{2})x}$

14、

$y'' + 2k^2y' + k^4y = 0$

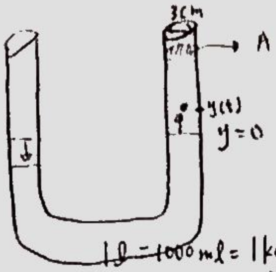
$\lambda^2 + 2k^2\lambda + k^4 = 0$

$\lambda \quad + k^2$
$\lambda \quad + k^2$

$(\lambda + k^2)^2 = 0 \qquad \lambda = k^2$

$y = Ae^{-k^2x} + Bxe^{-k^2x}$ #

2-4.



$A = \pi (1.5)^2 \text{ cm}^2 = 2.25\pi \text{ cm}^2$

$1\ell = 1000 \text{ m}\ell = 1\text{kg}$

$$F = 2.25\pi \text{ cm}^2 \cdot 2y(t)\, g = 1\text{kg}\, \ddot{y}(t),$$

$$= -4.5\pi \times 10^{-4} \text{m}^2\, y(t)\, g = 1\text{kg}\, \ddot{y}(t)$$

SI

$$4.5\pi \times 10^{-4} g\, y = \ddot{y}$$

$$y = e^{i\sqrt{4.5\pi \times 10^{-4} g}\, t} + e^{-i\sqrt{4.5\pi \times 10^{-4} g}\, t}$$

$$\omega = \sqrt{4.5\pi \times 10^{-4} g}$$

$$\simeq 10^{-2} \times \sqrt{4.5 \times 3.14 \times 9.8}$$

$$\simeq 0.118$$

18、

(10) under damping $y(t) = e^{-\alpha t}(A\cos\omega t + B\sin\omega t)$

If at t_1 has maximum, the next maximum will at $t_1 + \frac{2\pi}{\omega} = t_2$

$\because \cos\omega t_2 = \cos\omega t_1$ & $\sin\omega t_2 = \sin\omega t_1$,

$$\frac{y(t_1)}{y(t_2)} = \frac{e^{-\alpha t_1}\cancel{(A\cos\omega t_2 + B\sin\omega t_2)}}{e^{-\alpha t_2}\cancel{(A\cos\omega t_1 + B\sin\omega t_1)}} = e^{-\alpha(t_1-t_2)} = e^{\frac{2\alpha\pi}{\omega}}$$

take natural log, $\Delta = \frac{2\alpha\pi}{\omega}$ ※

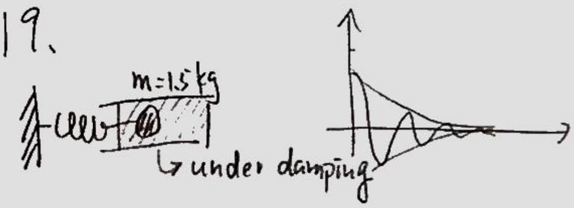
For $y'' + 4y' + 13y = 0$, $\lambda^2 + 4\lambda + 13 = 0$,

$$\lambda = \frac{-4 \pm \sqrt{16-52}}{2} = \frac{-4\pm\sqrt{-36}}{2} = \frac{-4\pm 6i}{2} = -2\pm 3i$$

$$y(t) = e^{-2t}(A\cos 3t + B\sin 3t)$$

$$\Delta = \frac{2\cdot 2\cdot \pi}{3} = \frac{4\pi}{3} ※$$

19、



① $\frac{2\pi}{\omega} = 3$ sec. $\xrightarrow{\text{from ⑧}}$ $\omega = \frac{2\pi}{3}$

② $e^{-\frac{2\pi}{\omega}\cdot 15\cdot\alpha} = \frac{1}{2}$, $e^{-\frac{2\pi}{2\pi/3}\cdot 15\cdot\alpha} = \frac{1}{2}$, $e^{-45\alpha} = \frac{1}{2}$,

$45\alpha = \ln 2$, $\alpha = \frac{1}{45}\ln 2$

$\lambda = \frac{-1}{45}\ln 2 \pm \frac{2\pi i}{3}$, $45\lambda = -\ln 2 \pm 30\pi i$,

$45\lambda + \ln 2 = \pm 30\pi i$, $(45\lambda + \ln 2)^2 = -900\pi^2$, $2025\lambda^2 + 90\lambda\ln 2 + \ln 2\ln 2 = -900\pi^2$

$\lambda^2 + \boxed{\frac{90\ln 2}{2025}}\lambda + 900\pi^2 + (\ln 2)^2 = 0$ the ODE is $y'' + \frac{2\ln 2}{45}y' + 900\pi^2 + (\ln 2)^2 = 0$.

If $my'' + cy' + ky = 0$, c = damping constant,

→ Damping constant $= \overset{0.1}{\cancel{1.5}} \times \frac{2\ln 2}{\underset{3}{\cancel{45}}} = \frac{\ln 2}{15}$ ※

2.5

Euler - Cauchy equations

$$x^2y'' + axy' + by = 0$$

Let $y = x^m$, $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

$$\cancel{x^2}\, m(m-1)\cancel{x^{m-2}} + a\cancel{x} \cdot m\cancel{x^{m-1}} + b\cancel{x^m} = 0$$

$$m(m-1) + am + b = 0$$

$$m^2 + (a-1)m + b = 0$$

$$m = \frac{(1-a) \pm \sqrt{(1-a)^2 - 4b}}{2}$$

$\rightarrow$ $m = \alpha$ or $\beta \in \mathbb{R}$ & $\alpha \neq \beta$:

$$y = c_1 x^\alpha + c_2 x^\beta$$

$\rightarrow$ $m = \alpha$:

$$y = (c_1 + c_2 \ln x)\, x^\alpha$$

$\rightarrow$ $m = \alpha \pm \beta i$:

$$y = c_1 x^{\alpha+\beta i} + c_2 x^{\alpha - \beta i}$$

$$= x^\alpha (c_1 x^{+\beta i} + c_2 x^{-\beta i})$$

$$= x^\alpha (c_1 e^{\beta i \ln x} + c_2 e^{-\beta i \ln x})$$

$$= x^\alpha (c_1' \sin\beta \ln x + c_2' \cos\beta \ln x)$$

2-5

14.

$x^2y'' + xy' + 9y = 0 \quad y(1) = 0,\ y'(1) = 2.5$

Let $y = x^m$, $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

$m(m-1) + m + 9 = 0$

$m^2 - \cancel{m} + \cancel{m} + 9 = 0$, $m^2 = -9$, $m = \pm 3i$

$y = C_1 x^{3i} + C_2 x^{-3i}$

$= C_1 e^{3i\ln x} + C_2 e^{-3i\ln x}$

$= C_1' \sin 3\ln x + C_2' \cos 3\ln x$

$y(1) = 0$, $C_1' \cancel{\sin 0}^{0} + C_2' \cos 0 = 0$, $C_2' = 0$

$y'(1) = C_1' \frac{3}{x} \cos 3\ln x - C_2' \frac{3}{x} \sin 3\ln x$

$= 3C_1' \cos 0 - 3C_2' \sin 0 = 2.5$

$3C_1' = 2.5$, $C_1' = \frac{5}{6}$ #

$y = \frac{5}{6} \sin 3\ln x$ #