Method.

Special functions.

Ch5. Series solution of ODEs. (Special functions)

Power Series method

Let $y(x)=\sum_{m=0}^{\infty} a_m x^m$

代回 ODE. 解各 a_m.

Hw. 5.1.6.

Extended Power Series method : Frobenius Method

解 $y''+\frac{b(x)}{x}y'+\frac{c(x)}{x^2}y=0$.

解 Bessel.

Let $y(x)=\sum_{m=0}^{\infty} a_m x^{r+m}$ (assume $a_0\neq 0$)

代回 ODE. 解 r. (看最低階項)

If $r=r_1$ or r_2

① 代入 r_1. r_2 檢查 a_0 是否 $=0$. (是的話代表使 $a_0=0$ 的不是函數解的基底)

② r_1-r_2 是否非整數? (是的話代表 r_1. r_2 不是完整基底)

如果只有一根基底可用, Let $y_2=u(x)y_1$, 代回 ODE 解 $u(x)$ & y_2

$y=c_1y_1+c_2y_2$.

Legendre's Eqn.

Solve

$(1-x^2)y''-2xy'+n(n+1)y=0$

Solution

$$P_n(x)=\sum_{m=0}^{M}(-1)^m\frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}x^{n-2m}$$

$M=\frac{n}{2}$ or $\frac{n-1}{2}$, $M=0,1,2,3\cdots$

Bessel's Eqn.

Solve.

$x^2y''+xy'+(x^2-v^2)y=0$

Hw. 5.4.3

Solution

$$J_v(x)=\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+v}}{2^{2m+v}m!(v+m)!}$$

$$Y_v(x)=\frac{1}{\sin v\pi}[J_v(x)\cos v\pi-J_{-v}(x)]$$

$$y(x)=c_1J_v(x)+c_2Y_v(x).$$

Property

$[x^vJ_v(x)]'=x^vJ_{v-1}(x)$

$[x^{-v}J_v(x)]'=-x^{-v}J_{v+1}(x)$ Hw. 5.4.22

$J_{v-1}(x)+J_{v+1}(x)=\frac{2v}{x}J_v(x)$

$J_{v-1}(x)-J_{v+1}(x)=2J_v'(x)$

Review : Radius of Convergence

1. $\frac{1}{R}=\lim_{m\to\infty}|a_m|^{1/m}$

Hw. 5.1.3.

2. $\frac{1}{R}=\lim_{m\to\infty}\frac{a_{m+1}}{a_m}$.

Try to solve $x^2y'' + xy' + (x^2 - \nu^2)y = 0$.

Let $y = \sum_{m=0}^{\infty} a_m x^{\boxed{}}$

$y' = \underline{\qquad\qquad\qquad}$, $y'' = \underline{\qquad\qquad\qquad}$

代回 ODE：

$x^2 \underline{\qquad\qquad\qquad} + x \underline{\qquad\qquad\qquad} + x^2 \underline{\qquad\qquad\qquad} - \nu^2 \underline{\qquad\qquad\qquad} = 0$

找到最低階：

当 $r = \nu$:

$a_0 = a_0$, $a_1 = \underline{\qquad}$, $a_2 = \underline{\qquad}$ $a_3 = \underline{\qquad}$ $\cdots$ $a_{m\,(m=2,4,6\cdots)} = \underline{\qquad\qquad}$

变换 $m' = \frac{1}{2}m$, $m = 2m$. $a_{2m'} = \underline{\qquad\qquad\qquad}$

Let $a_0 = \frac{1}{2^\nu \nu!}$, $a_{2m'} = \underline{\qquad\qquad\qquad}$

$J_\nu(x) = \underline{\qquad\qquad\qquad\qquad}$